Exponents

Raghuram is a banker, he makes money by lending money. He had lent Rs 100 to his friend Rajan, as Rajan had to buy some sugar, to make sweets and sell, ahead of the Dassehra festival. Rajan returned Rs120 to Raghuram, after the festival, which eventually fetched Raghuram a profit of Rs20.

However, Raghuram decided to learn Mathematics and find better ways of lending money, so that he could make more money. Raghuram then came out with a concept called "Simple Interest" – he would just collect 10%(extra amount, also called interest) of his money every month- if Rajan took Rs100 and if he returns it after one month, he would have to pay the lent amount (Rs100) along with an interest which is 10% of Rs 100 (which equals Rs10).

Rajan had a loss during the festival and repaid the money only one year later, although he had agreed to pay the money back in one month. So, Raghuram collected an interest of Rs120 (towards the twelve months of the year) and so a total amount of Rs 220 (Rs100 + an interest of Rs120), during the end of the year. However, Raghuram's friend Pascal asked him an interesting question. Raghuram realized that he was actually losing money.

Pascal gave him a simple example. He said, you gave Rs100 to Raghuram on Jan 1st. You should have got Rs110 (Rs100 + an interest of rs10 for one month) on Jan 31st. And if that doesn't happen and say he returned the money in February, you would have collected Rs 120 (Rs100 + an interest of Rs20 for two months). Pascal showed Raghuram that he made a loss when he collected Rs120 after two months. Pascal explained, had Rajan returned Rs110 on Jan 31st, you would have lent Rs 110 on Feb 1st and collected Rs121 on Feb 28th (Rs110 actual amount + an interest of Rs11 which is 10% of 110). This meant that Raghuram had a loss of Rs1. You might Rs1 loss may be very small, but then think of the loss after twelve months- is it just Rs1 *12 = Rs12? Let us calculate and check it out!

Jan 1st: Rajan gets Rs100	**Jan 31st:** Rajan returns Rs110 (Rs100 + 10% of 100)
Feb 1st: Raghuram lends Rs110	**Feb 28th:** Rajan returns Rs121 (Rs110 + 10% of 110)
March 1st: Raghuram lends Rs121	**March 31st:** Rajan returns Rs133.1 (Rs121 + 10% of 121)
April 1st: Raghuram lends Rs133.1	**April 30th:** Rajan returns Rs146.41
May 1st: Raghuram lends Rs146.41	**May 31st:** Rajan returns Rs161.051
June 1st: Raghuram lends Rs161.051	**June 30th:** Rajan returns Rs177.1561
July 1st: Raghuram lends Rs177.1561	**July 31st:** Rajan returns Rs194.87171
August 1st: Raghuram lends Rs194.87171	**August 31st:** Rajan returns Rs214.35881
September 1st: Raghuram lends Rs214.35881	**September 31st:** Rajan returns Rs235.7947691
October 1st: Raghuram lends Rs235.7947691	**October 31st:** Rajan returns Rs259.37424601
November 1st: Raghuram lends Rs259.37424601	**November 31st:** Rajan returns Rs285.311670611
December 1st: Raghuram lends Rs285.311670611	**December 31st:** Rajan returns Rs313.84284

As per this logic shown by Pascal, Rajan had to pay Rs313.84284 after one year, but he had actually paid only Rs220. So Raghuram got a loss of Rs93.84284! Isn't that a huge loss- almost half of the money Rajan paid him or almost one third of the money which Rajan had to pay? In this method, Raghuram was missing out on the interest on the interest, interest on the interest on the interest and so on!

Date	Money collected	Pascal Method	Loss	Observation
Jan 31st	Rs110	Rs 110	Rs 0	= 100 * 1.1
Feb 28th	Rs120	Rs121	Rs 1	= 100 * 1.1 * 1.1
March 31st	Rs 130	Rs133.1	Rs 3.1	= 100 * 3 1.1s
April 30th	Rs 140	Rs146.41	Rs 6.41	=100 *4.. 1.1s
May 31st	Rs 150	Rs161.051	Rs 11.051	=100 *5.. 1.1s
June 30th	Rs 160	Rs177.1561	Rs 17.1561	=100 *6.. 1.1s
July 31st	Rs 170	Rs194.87171	Rs 24.87171	=100 *7.. 1.1s
August 31st	Rs 180	Rs214.35881	Rs 34.35881	=100 *8.. 1.1s
September 30th	Rs 190	Rs235.7947691	Rs 45.794691	=100 *9.. 1.1s
October 31st	Rs 200	Rs259.37424601	Rs 59/37424601	=100 *10.. 1.1s
November 30th	Rs 210	Rs285.311670611	Rs 75.311670611	=100 *11.. 1.1s
December 31st	Rs 220	Rs313.84284	Rs 93.84284	=100 *12.. 1.1s

This can be visualized this way- money collected by Raghuram is

Month End Collection = Month Start Amount + Interest

= Month Start Amount*1 + 0.1 *Month Start Amount (As a*1=1)

= Month Start Amount *(1 + 0.1) (Distributive law – a*(b+c)=a*b+a*c)

= Month Start Amount * 1.1

For example, Month End Collection (June) = Month Start (June) Amount * 1.1 = Rs 161.051*1.1= Rs177.1561 (this agreed with our table prediction on June 30th).

First Month: So, amount on Jan 31st = 100 * 1.1 (using the above formula)
Second Month: Likewise, amount on Feb 28th = Jan 31st *1.1 = (100 * 1.1)*1.1
Third Month: Amount on March 31st = Feb 28th Amount * 1.1 = (100*1.1*1.1)*1.1= 100 *1.1*1.1*1.1
So in each step, one 1.1 gets multiplied
Twelfth Month: Amount = 100 * (1.1*1.1*1.1*1.1*1.1*1.1*1.1*1.1*1.1*1.1*1.1*1.1)

Initially the loss wasn't much, but eventually the loss kept on piling! And let's see something interesting. The Pascal method amount after 6 months is basically 100 * 1.1 * 1.1 * 1.1 * 1.1 * 1.1 * 1.1. Likewise, if you want the amount after 12months, you multiply 100 with 12- 1.1s. Why don't we get a better symbol to convey this idea?

11*11 is written as 11^2 , 11*11*11 is written as 11^3 and likewise 11 multiplied 10 times is 11^{10}. These are called exponents and they grow really really fast! One of the most motivational message ever is written below, have a look.

1.01^{365} = 37.8 (If you improve by 0.01, consistently over 365 days, you reach from 1 point to 37.8 points)
0.99^{365}=0.03 (if you lose 0.01 points every-day, you are left with 0.03 points at the end of the year)

So when we say Ravi is progressing exponentially, it means he is growing really really fast. Or one could say you're on an exponential decline, in case you are just collapsing.

Questions for discussion

Q1 How did Raghuram earn money by lending money? What is interest? How is interest equivalent to money earnt? Can Raghuram make a lot of money by just lending money?

Q2 Why did Raghuram collect Rs120 after two months? Why did he collect Rs220 after two years?

Q3 Why did Raghuram want to join in a Mathematics class? Was he correct in doing so?

Q4 Why did Raghuram get Rs121 at the end of two months, as per Pascal method? The interest per month is Rs10 (10% of Rs100), why is he collecting 10% of 110 during the second month?

Q5 If you are Raghuram and if you'd like to make a lot of money, would you add 10% on the month end due or would you collect 10% per month * total number of months?

Q6 Why did Raghuram collect Rs133.1 at the end of third month? Why is he collect 10% of 121 as the interest during the third month?

Q7 Why is one 1.1 getting multiplied after each step?

Q8 Why do you think the amount is growing as $100 * 1.1^n$ after n months? (For instance it is $100* 1.1^6$after 6 months). Why is amount at the end of third month equal to second month amount * 1.1?

Q9 What is interest on interest on interest? How much interest does Raghuram get on Rs100 (by charging 10%)? How much interest does Raghuram by charging 10% on that interest? Is it equal to the loss in that month? So why is Raghuram getting a loss?

Q10 1.01^{365}=37.8 is interpreted as an improvement of 0.01 over 365 days. How is it equivalent to a daily improvement? Can 0.01 be seen like an interest? 0.99^{365}=0.03 is interpreted as a loss of 0.01 every day, why? How is this motivational? What do you mean by exponential growth/decay?

Commentary
Consider the three statements

Statement1: Ravi ate four apples and kept three apples in his box [English]
Statement2: 7 -4=3 [Arithmetic]
Statement3: Total – Ate = 3 [Algebra]

Which of the three statements was the easiest to understand? Why is English easier to understand? If you learnt French after learning English and if you don't use French much, why would English be easier to understand? Why is Arithmetic easier to understand compared to Algebra?

So if too many mathematical symbols are hard to understand, would it make sense to translate the equations/steps into English? Would English be easier to understand? So what would you do if this explanation of interest wasn't clear?

Does Mathematics add to your understanding of Language? When you use the phrase exponential growth/decay, what do you visualize? What is the difference after solving a few problems of exponents? How does Mathematics change the way you describe nature?